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# IFToMM Benchmark Problem Wheel on Tilting Table

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# CONTENTS

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<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	System Description . . . . .	1
<b>2</b>	<b>Analysis</b>	<b>3</b>
2.1	Eigenvalue Analysis . . . . .	3
2.2	Frequency Response Plots . . . . .	5
<b>A</b>	<b>Equations of Motion</b>	<b>6</b>

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# LIST OF FIGURES

---

2.1 Frequency response: Table actuator . . . . . 5

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# LIST OF TABLES

---

1.1	Body CG Locations and Mass . . . . .	1
1.2	Body Inertia Properties . . . . .	1
1.3	Connection Location and Direction . . . . .	2
2.1	Eigenvalues . . . . .	3
2.2	Eigenvalue Analysis . . . . .	3

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# CHAPTER 1

## INTRODUCTION

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This report provides the results of the analysis of the IFToMM four bar linkage benchmark problem using the EoM software produced by the University of Windsor Vehicle Dynamics and Control research group. The problem consists of two rigid bodies, a wheel rolling without slip on a table free to tilt. The properties are summarized below. The problem is modified slightly by defining a torque on the table as the system input and the angular motion of the wheel as the system output. Note that there may be some round-off in the data describing the system properties. Please see the problem definition document for more precise values.

### 1.1 System Description

The properties of the bodies are given in Tables 1.1 and 1.2. The properties of the connections are given in Table 1.3.

**Table 1.1:** Body CG Locations and Mass

No.	Body Name	Location [m]	Mass [kg]
1	Wheel	0.000, 0.000, 0.250	1.000
2	Table	0.000, 0.000, 0.000	1.000

**Table 1.2:** Body Inertia Properties

No.	Body Name	Inertia [kg·m <sup>2</sup> ] ( $I_{xx}$ , $I_{yy}$ , $I_{zz}$ ; $I_{xy}$ , $I_{yz}$ , $I_{zx}$ )
1	Wheel	0.016, 0.031, 0.016; 0.000, 0.000, 0.000
2	Table	0.083, 0.083, 0.167; 0.000, 0.000, 0.000

Note: inertias are defined as the positive integral over the body, e.g.,  $I_{xy} = + \int r_x r_y dm$ .

**Table 1.3:** Connection Location and Direction

No.	Connection Name	Location [m]	Unit Axis
1	Wheel contact	0.000, 0.000, 0.000	0.000, 1.000, 0.000
2	Table joint	0.000, 0.000, 0.000	0.000, 1.000, 0.000

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## CHAPTER 2

# ANALYSIS

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The EoM software automatically conducts a linear analysis after producing the linearized equations of motion. The results listed below show strong agreement with the IFTOMM results.

### 2.1 Eigenvalue Analysis

The eigenvalue properties are given in Tables 2.1 and 2.2.

**Table 2.1:** Eigenvalues

No.	Real [rad/s]	Imaginary [rad/s]	Real [Hz]	Imaginary [Hz]
1	$-9.5135819144 \times 10^{-16}$	$6.7913418692 \times 10^0$	$-1.5141335882 \times 10^{-16}$	$1.0808756287 \times 10^0$
2	$-9.5135819144 \times 10^{-16}$	$-6.7913418692 \times 10^0$	$-1.5141335882 \times 10^{-16}$	$-1.0808756287 \times 10^0$
3	$4.0856241120 \times 10^0$	$0.0000000000 \times 10^0$	$6.5024727304 \times 10^{-1}$	$0.0000000000 \times 10^0$
4	$-4.0856241120 \times 10^0$	$0.0000000000 \times 10^0$	$-6.5024727304 \times 10^{-1}$	$0.0000000000 \times 10^0$

Note: oscillatory roots appear as complex conjugates.

**Table 2.2:** Eigenvalue Analysis

No.	Frequency ( $\omega_n$ ) [Hz]	Damping Ratio ( $\zeta$ )	Time Constant ( $\tau$ ) [s]	Wavelength ( $\lambda$ ) [s]
1	$1.0808756287 \times 10^0$	$1.4008397895 \times 10^{-16}$	$1.0511288062 \times 10^{15}$	$9.2517582360 \times 10^{-1}$
2	$1.0808756287 \times 10^0$	$1.4008397895 \times 10^{-16}$	$1.0511288062 \times 10^{15}$	$9.2517582360 \times 10^{-1}$
3	–	–	$-2.4476064674 \times 10^{-1}$	–
4	–	–	$2.4476064674 \times 10^{-1}$	–

Notes: a) oscillatory roots are listed twice, b) negative time constants denote unstable roots.

There are 2 degrees of freedom.

There are 1 oscillatory modes, 2 damped modes, 1 unstable modes, and 0 rigid body modes.



## 2.2 Frequency Response Plots

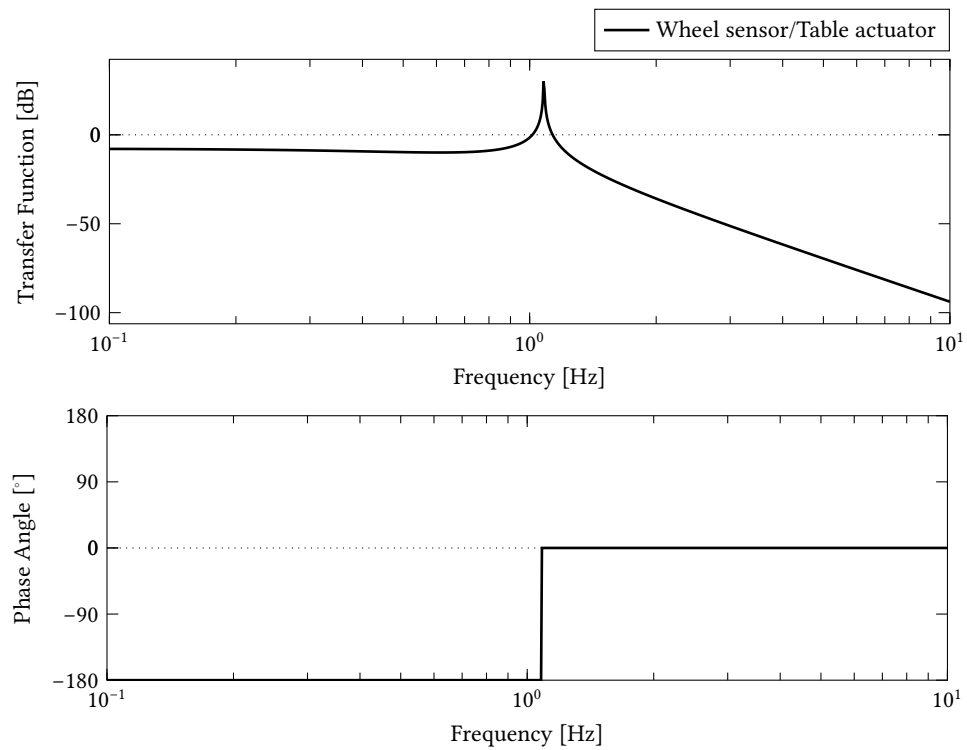


Figure 2.1: Frequency response: Table actuator

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## APPENDIX A

# EQUATIONS OF MOTION

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The equations of motion are of the form

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & -\mathbf{G} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{w}} \\ \dot{\mathbf{u}} \end{Bmatrix} + \begin{bmatrix} \mathbf{V} & -\mathbf{I} & \mathbf{0} \\ \mathbf{K} & \mathbf{L} & -\mathbf{F} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{p} \\ \mathbf{w} \\ \mathbf{u} \end{Bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \end{bmatrix} \{\mathbf{u}\}$$

The mass matrix of the system is

Row	Column	Value	Row	Column	Value
1	1	$1.00000000 \times 10^0$	7	7	$1.00000000 \times 10^0$
2	2	$1.00000000 \times 10^0$	8	8	$1.00000000 \times 10^0$
3	3	$1.00000000 \times 10^0$	9	9	$1.00000000 \times 10^0$
4	4	$1.56250000 \times 10^{-2}$	10	10	$8.33333333 \times 10^{-2}$
5	5	$3.12500000 \times 10^{-2}$	11	11	$8.33333333 \times 10^{-2}$
6	6	$1.56250000 \times 10^{-2}$	12	12	$1.66666667 \times 10^{-1}$

The damping matrix is

Row	Column	Value	Row	Column	Value
1	1	$0.00000000 \times 10^0$	–	–	–

The stiffness matrix is

Row	Column	Value	Row	Column	Value
11	1	$-9.81000000 \times 10^0$	11	7	$9.81000000 \times 10^0$
10	2	$9.81000000 \times 10^0$	10	8	$-9.81000000 \times 10^0$
4	4	$-2.45250000 \times 10^0$	5	11	$-2.45250000 \times 10^0$
10	4	$2.45250000 \times 10^0$	11	11	$2.45250000 \times 10^0$

The velocity matrix is

Row	Column	Value	Row	Column	Value
1	1	$0.00000000 \times 10^0$	–	–	–

The input force matrix is

Row	Column	Value	Row	Column	Value
11	1	$-1.00000000 \times 10^0$	–	–	–

The input force rate matrix is

Row	Column	Value	Row	Column	Value
1	1	$0.00000000 \times 10^0$	–	–	–

The system is subject to constraints

$$\begin{bmatrix} \mathbf{J}_h & \mathbf{0} & \mathbf{0} \\ -\mathbf{J}_h \mathbf{V} & \mathbf{J}_h & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{nh} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}} & \mathbf{p} \\ \dot{\mathbf{w}} & \mathbf{w} \\ \dot{\mathbf{u}} & \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Row	Column	Value	Row	Column	Value
1	1	$1.00000000 \times 10^0$	11	13	$1.00000000 \times 10^0$
2	2	$1.00000000 \times 10^0$	12	14	$1.00000000 \times 10^0$
3	3	$1.00000000 \times 10^0$	13	15	$1.00000000 \times 10^0$
2	4	$2.50000000 \times 10^{-1}$	12	16	$2.50000000 \times 10^{-1}$
4	4	$-1.00000000 \times 10^0$	14	16	$-1.00000000 \times 10^0$
1	5	$-2.50000000 \times 10^{-1}$	11	17	$-2.50000000 \times 10^{-1}$
5	6	$1.00000000 \times 10^0$	15	18	$1.00000000 \times 10^0$
1	7	$-1.00000000 \times 10^0$	11	19	$-1.00000000 \times 10^0$
6	7	$1.00000000 \times 10^0$	16	19	$1.00000000 \times 10^0$
2	8	$-1.00000000 \times 10^0$	12	20	$-1.00000000 \times 10^0$
7	8	$1.00000000 \times 10^0$	17	20	$1.00000000 \times 10^0$
3	9	$-1.00000000 \times 10^0$	13	21	$-1.00000000 \times 10^0$
8	9	$1.00000000 \times 10^0$	18	21	$1.00000000 \times 10^0$
4	10	$1.00000000 \times 10^0$	14	22	$1.00000000 \times 10^0$
9	10	$-1.00000000 \times 10^0$	19	22	$-1.00000000 \times 10^0$
5	12	$-1.00000000 \times 10^0$	15	24	$-1.00000000 \times 10^0$
10	12	$1.00000000 \times 10^0$	20	24	$1.00000000 \times 10^0$

The full state space equations:

$$\begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{x}} \\ \mathbf{y} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \mathbf{u} \end{Bmatrix}$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \left[ \begin{array}{cccccc} -2.17070941 \times 10^{-1} & -8.68249441 \times 10^{-1} & -3.87913317 \times 10^{-1} & 1.50953252 \times 10^0 & 0.00000000 \times 10^0 & 0.00000000 \times 10^0 \\ -7.59466287 \times 10^{-1} & -6.26744518 \times 10^{-1} & -6.31797333 \times 10^{-1} & -3.05409239 \times 10^{-2} & 0.00000000 \times 10^0 & 0.00000000 \times 10^0 \\ 2.40066418 \times 10^0 & -1.68731881 \times 10^{-1} & 0.00000000 \times 10^0 & -2.42570492 \times 10^0 & -1.00000000 \times 10^0 & 0.00000000 \times 10^0 \\ -8.40479654 \times 10^{-1} & -1.09193438 \times 10^0 & 6.71085232 \times 10^{-1} & 8.43815459 \times 10^{-1} & 0.00000000 \times 10^0 & 0.00000000 \times 10^0 \\ 0.00000000 \times 10^0 & 0.00000000 \times 10^0 & 0.00000000 \times 10^0 & 0.00000000 \times 10^0 & -1.00000000 \times 10^0 & 1.00000000 \times 10^0 \\ \hline -5.90950774 \times 10^{-1} & 7.00597285 \times 10^{-1} & 0.00000000 \times 10^0 & 3.17989148 \times 10^{-1} & 0.00000000 \times 10^0 & 0.00000000 \times 10^0 \end{array} \right]$$

$$\mathbf{E} = \left[ \begin{array}{ccccc} 5.63744280 \times 10^{-1} & -1.77622765 \times 10^{-1} & 0.00000000 \times 10^0 & -4.19396791 \times 10^{-1} & 0.00000000 \times 10^0 \\ -1.77622765 \times 10^{-1} & 9.27680383 \times 10^{-1} & 0.00000000 \times 10^0 & -1.70758604 \times 10^{-1} & 0.00000000 \times 10^0 \\ 0.00000000 \times 10^0 & 0.00000000 \times 10^0 & 8.33333333 \times 10^{-2} & 0.00000000 \times 10^0 & 0.00000000 \times 10^0 \\ -4.19396791 \times 10^{-1} & -1.70758604 \times 10^{-1} & 0.00000000 \times 10^0 & 5.96810631 \times 10^{-1} & 0.00000000 \times 10^0 \\ 0.00000000 \times 10^0 & 0.00000000 \times 10^0 & 0.00000000 \times 10^0 & 0.00000000 \times 10^0 & 0.00000000 \times 10^0 \end{array} \right]$$

The reduced state space equations:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \left[ \begin{array}{cccc|c} -6.81410694 \times 10^0 & -1.16128915 \times 10^1 & -1.55165327 \times 10^0 & 6.46116152 \times 10^0 & 0.00000000 \times 10^0 \\ -3.44546838 \times 10^0 & -5.00145629 \times 10^0 & -2.52718933 \times 10^0 & 2.30810470 \times 10^0 & 0.00000000 \times 10^0 \\ 7.20199255 \times 10^0 & -5.06195642 \times 10^{-1} & 0.00000000 \times 10^0 & -3.63855737 \times 10^0 & -1.20000000 \times 10^1 \\ -1.43651516 \times 10^1 & -2.28427081 \times 10^1 & 5.36868185 \times 10^0 & 1.18155632 \times 10^1 & 0.00000000 \times 10^0 \\ \hline -1.47737693 \times 10^{-1} & 1.75149321 \times 10^{-1} & 0.00000000 \times 10^0 & 3.97486435 \times 10^{-2} & 0.00000000 \times 10^0 \end{array} \right]$$